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A NONASYMPTOTIC CRITERION FOR THE EVALUATION OF AUTOMOBILE BONUS SYSTEMS

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Summary

A new criterion for the evaluation of automobile bonus systems is proposed. It states that a bonus system should be constructed such as to minimize a weighted average of the expected squared rating errors in various insurance periods. The criterion generalizes an asymptotic criterion given earlier by Norberg in 1976. In addition, the new nonasymptotic criterion makes it possible to discuss various short term aspects such as the optimal choice of starting class and the time heterogeneity of risks. Our treatment is illustrated by examples with numerical results.

1. Introduction

The model framework of modern credibility theory, with its fundamental notion of randomly distributed risk parameters, was employed in an analysis of automobile bonus systems by Pesonen as early as in 1963. He suggested that each policy in any bonus class j should be charged a premium which is the expected claim amount per year of an infinitely old policy in this class. This determines a bonus scale (a set of class premiums) for a given set of bonus rules governing the inter-class transitions of the policies.

One of the present authors (Norberg 1976) later completed this line of ideas by showing that Pesonen's bonus scale minimizes the expected squared rating error (i.e. the difference between the true pure premium and the premium actually paid) for a randomly chosen policy in the stationary state, i.e. a policy which has been subjected to the bonus rules during infinitely many insurance periods. Furthermore, this asymptotic criterion extends the previous theory by opening for the comparison not only of different bonus scales under fixed rules, but also for the comparison of different sets of bonus rules. Norberg used the criterion to assess the effects of selected modifications of the then current Norwegian bonus rules.

One may object to the asymptotic criterion on the ground that policies are in force only during a limited number of insurance periods. If a majority of the policies are far from the stationary state, it seems desirable to modify the criterion so as to take into account the rating error for new policies and for policies of a moderate age as well. It is the main purpose of the present paper to propose such a modification. It is based on the simple idea that the performance of a bonus system can be measured by a weighted average of the expected squared rating errors for selected insurance periods.

In a natural way the results based on this new criterion generalize those formerly obtained for the purely asymptotic criterion: To a fixed set of bonus rules there exists a unique, optimal bonus scale, and competing sets of bonus rules may be compared by means of a simple measure of efficiency.

The new, nonasymptotic criterion permits a rational treatment of various short term aspects of practical importance which could not be analysed by means of the asymptotic criterion, such as the optimal choice of starting class and the time heterogeneity of risks. For instance numerical results given below indicate that the Norwegian bonus system could be improved by choosing a higher bonus class for new entrants in the case of time homogeneity. Our conclusions appear to be fairly robust to changes in the weights used in the weighted average criterion in the case considered by us.

These weights may be given various interpretations. We offer two, viz. that the weights (i) are discount factors corresponding to a given rate of interest, or (ii) express the age distribution of the policies in the portfolio.

The presentation is divided into sections as follows. Bonus systems are formally defined in Section 2. Section 3 presents the basic elements of the stochastic model. The results of Norberg (1976) are briefly reviewed in Section 4, and the nonasymptotic criterion is introduced and examined there. Section 5 contains a discussion of our results as well as numerical illustrations.

2. Definition of a bonus system

To provide a complete setting, we give the following extract from Norberg (1976), whose paper should be consulted for fuller details and examples.

The elements defining a bonus system are

- insurance periods of a fixed and equal length;
- bonus classes numbered from 1 to K ; a policy stays in one class throughout each insurance period;
- the bonus scale, which is the vector $\underset{\sim}{\pi} = \{\pi(1),\ldots,\pi(K)\}$ of single-period premiums for policies in classes $1,\ldots,K$;
- the bonus rules, R , which specify an initial class, k ,in which all new policies start (in period 1), and transition rules which determine the class of a policy in any period as a function of its bonus class and number of claims in the preceding period.

The transition rules are represented by a $K\times K$ - matrix $\underset{\sim}{T}$, whose entry T_{ij} in row i and column j is the set of claim numbers leading from class i to class j . The bonus rules can be given as the pair $R = (\underset{\sim}{T},k)$, and the bonus system is the triplet $S = (\underset{\sim}{T},k,\underset{\sim}{\pi})$, when a fixed period length is understood.

Our terminology differs slightly from that of Norberg (1976), who spoke of $\underset{\sim}{T}$ as the bonus rules. Since we take an interest in the finite time properties of bonus systems, the initial class becomes a significant part of the rules governing the moves of the policies. Hence we distinguish between the transition rules $\underset{\sim}{T}$ and the complete set R of bonus rules.

3. General assumptions about the risk process

Consider a risk drawn at random from a group of risks, in the present context motor vehicle insurance policies. Let M_n denote the number of claims reported by the risk in period n, and let $Y_{n1}, Y_{n2}, \ldots$ be the severities of these single claims, recorded in chronological order. For convenience we also define $Y_{n0} = 0$. The total claim amount of the risk in year n is

$$X_n = \sum_{j=0}^{M_n} Y_{nj} .$$

Although the group is virtually homogeneous with respect to some basic, observable risk characteristics, there will remain risk differentials between the policies due to unobservable risk characteristics. Such latent risk characteristics are represented by a <u>risk parameter</u> θ. The value of θ of the current, randomly picked risk is regarded as the outcome of an unobservable random element Θ. The distribution $U(\cdot)$ of Θ represents the risk structure of the group.

The probability distribution of the claim numbers and severities depends on the value of the risk parameter. If Θ were known to have value θ, one would charge the <u>true pure premium</u> in period n, viz., $E_\theta X_n$. (The subscript θ signifies conditioning w.r.t. $\Theta = \theta$.) At the outset, Θ is totally unknown. As time goes on its value is reflected by the risk performance of the policy. This fact makes individual experience rating possible, whereby $E_\Theta X_n$ is approximated by a function of the observed $\{M_i\}$ and $\{Y_{ij}\}$.

The bonus system defined in Section 2 is an example of an experience rating plan. Let $Z_{R,n}$ denote the bonus class in period n for the individual policy. It is determined by the bonus rules and by the claim numbers as follows: First $Z_{R,1} = k$. For $n \geq 1$, if $Z_{R,n} = i$ and $M_n \in T_{ij}$, then $Z_{R,n+1} = j$.

The conditional distribution of $Z_{R,n}$, given $\Theta = \theta$, is the row vector

$$\underset{\sim}{p}_{R,\theta}^{(n)} = \{p_{R,\theta}^{(n)}(1),\ldots,p_{R,\theta}^{(n)}(K)\} \tag{3.1}$$

with entries $p_{R,\theta}^{(n)}(j) = P_\theta(Z_{R,n}=j)$, $j = 1,\ldots,K$.
The unconditional distribution of $Z_{R,n}$ is

$$\underset{\sim}{p}_R^{(n)} = \int \underset{\sim}{p}_{R,\theta}^{(n)} \, d\,U(\theta) \, .$$

4. A non-asymptotic criterion for the choice of bonus elements

Under the bonus system $S = (\underset{\sim}{T},k,\underset{\sim}{\pi})$ the premium actually charged in period n is $\pi(Z_{R,n})$, and the expected squared rating error for this period is

$$Q_n(S) = E\{E_\Theta X_n - \pi(Z_{R,n})\}^2 = \int \sum_{j=1}^{K} \{E_\theta X_n - \pi(j)\}^2 \, p_{R,\theta}^{(n)}(j) \, d\,U(\theta). \tag{4.1}$$

Two bonus systems, S and $\tilde{S}$, may be compared by comparing the values of $Q_n(S)$ and $Q_n(\tilde{S})$. As a first step we note that for fixed R , $Q_n(S)$ is minimized by the choice $\underset{\sim}{\pi} = \underset{\sim}{\pi}_R^{(n)}$ given by

$$\pi_R^{(n)}(j) = E(E_\Theta X_n | Z_{R,n} = j) = \int E_\theta X_n p_{R,\theta}^{(n)}(j) \, d\,U(\theta)/p_R^{(n)}(j). \tag{4.2}$$

This follows from the well known fact that for general X and Z $E\{X-f(Z)\}^2$ attains its minimum when $f(Z) = E(X|Z)$. Thus the search for a bonus system which makes $Q_n(S)$ small, can be restricted to systems with a Q_n-optimal bonus scale determined by R as in (4.2). Hence the problem is reduced to choosing the bonus rules.

We introduce the Q_n-efficiency of the rules R,

$$e_n(R) = \sum_{j=1}^{K} \{\pi_R^{(n)}(j)\}^2 \, p_R^{(n)}(j) \, . \tag{4.3}$$

Let R and $\tilde{R}$ be two sets of bonus rules and let S and $\tilde{S}$ be the corresponding bonus systems with scales constructed in accordance with (4.2). Lemma 2.4 in Norberg (1976) states that $Q_n(S) < Q_n(\tilde{S})$ if and only if

$$e_n(R) > e_n(\tilde{R}) \, . \tag{4.4}$$

The criterion based on (4.1), or equivalently on (4.3), suffers from the disadvantage that its conclusions may depend on n. To circumvent this difficulty, Norberg (1976) proposed a criterion based on some asymptotic considerations, essentially the following ones. Suppose that for any R $(\Theta, X_n, Z_{R,n})$ converges in distribution to a random triplet, say $(\Theta, X_0, Z_{R,0})$, as $n \to \infty$. Assume, moreover, that the variables $(E_\Theta X_n)^2$, $n = 1,2,\ldots,$ are uniformly integrable. Let $Z_{R,0}$ have the conditional distribution $\underline{p}_{R,\theta}^{(0)}$ when $\Theta = \theta$, and define $Q_0(S)$ by letting $n = 0$ in (4.1). Then $Q_0(S)$ is finite, and

$$Q_n(S) \to Q_0(S) \qquad \text{as} \quad n \to \infty \, .$$

The random variables X_0 and $Z_{R,0}$ may be regarded as the claim amount per year and the bonus class of an (infinitely) old policy. The asymptotic criterion and the results derived from it, follows if n is set to 0 in relations (4.1) to (4.4).

The asymptotic criterion seems reasonable when a majority of the risks are close to the stationary state. In practice, however, risk portfolios will often have a substantial fraction of comparatively young policies. Then it is desirable to compare different bonus systems not only by means of Q_0 alone but also by means of Q_n for various finite n. This is achieved if the performance of S is measured by a weighted average of the form

$$\bar{Q}(S) = \sum_{n=0}^{\infty} w_n Q_n(S) . \tag{4.5}$$

The weights w_n are nonnegative and sum to 1. With $w_n = 1$ we are back to (4.1), and the choice of $w_0 = 1$ corresponds to the previous asymptotic criterion.

The following result generalizes formula (4.2) above as well as Norberg's (1976) Theorem 4.3.

<u>Theorem 1</u>. To any set of bonus rules R there exists an (almost surely) unique bonus scale $\bar{\pi}_R$ which is $\bar{Q}$-optimal in the sense that it minimizes $\bar{Q}(S)$ in (4.5). It is given by

$$\bar{\pi}_R(j) = \sum_{n=0}^{\infty} \pi_R^{(n)}(j) p_R(n|j) , \qquad j = 1,\dots,K , \tag{4.6}$$

where

$$p_R(n|j) = w_n p_R^{(n)}(j) / \sum_{\nu=0}^{\infty} w_\nu p_R^{(\nu)}(j). \tag{4.7}$$

<u>Proof</u>. Rewrite (4.5) as

$$\bar{Q}(S) = \sum_{n=0}^{\infty} w_n \int \sum_{j=1}^{K} \{E_\theta X_n - \pi(j)\}^2 p_{R,\theta}^{(n)}(j)\, dU(\theta) . \tag{4.5'}$$

By uniform integrability of the variables $E_\theta X_n$, differentiation of $\bar{Q}(S)$ with respect to $\pi(j)$, $j = 1,\dots,K$, can be performed inside of the summation and integration signs on the right hand side of (4.5'). Thus, for fixed rules R, $\bar{Q}(S)$ attains its minimum at the point $\bar{\pi}_R$ determined by the first order conditions

$$\sum_{n=0}^{\infty} w_n \int \sum_{j=1}^{K} \{E_\theta X_n - \bar{\pi}_R(j)\} p_{R,\theta}^{(n)}(j)\, dU(\theta) = 0\,, \qquad j = 1,\ldots,K\,.$$

The solution is

$$\bar{\pi}_R(j) = \sum_{n=0}^{\infty} w_n \int E_\theta X_n p_{R,\theta}^{(n)}(j)\, dU(\theta) / \sum_{n=0}^{\infty} w_n \int p_{R,\theta}^{(n)}(j)\, dU(\theta)\,,$$

$j = 1,\ldots,K$, which is easily rewritten as (4.6). □

Formula (4.6) may alternatively be derived by means of the following trick, which proves useful in establishing further results on optimal bonus systems. Introduce an integer-valued random variable N, independent of the risk process and with distribution

$$P(N = n) = w_n\,, \qquad n = 1,2,\ldots$$

Then $Q_n(S)$ may be expressed as

$$E[\{E_\theta X_N - \pi(Z_{R,N})\}^2 \mid N = n]\,,$$

and (4.5) can be cast in the compact form $\bar{Q}(S) = E\{E_\theta X_N - \pi(Z_{R,N})\}^2$, from which we immediately conclude that

$$\bar{\pi}_R(j) = E(E_\theta X_N \mid Z_{R,N} = j)\,. \tag{4.8}$$

With the interpretations $\pi_R^{(n)}(j) = E(E_\theta X_N | Z_{R,N} = j,\ N = n)$ and $p_R(n|j) = P(N = n | Z_{R,N} = j)$, formula (4.6) follows from (4.8) by the rule of iterated expectations.

We introduce

$$\bar{p}_R(j) = P(Z_{R,N} = j) = \sum_{n=0}^{\infty} w_n\, p_R^{(n)}(j)\,,$$

and define the $\bar{Q}$-efficiency of R to be

$$\bar{e}(R) = \sum_{j=1}^{K} \{\bar{\pi}_R(j)\}^2 \bar{p}_R(j) . \tag{4.9}$$

The proof of Lemma 2.4 in Norberg (1976) also serves to establish the following result.

Theorem 2. Let R and $\tilde{R}$ be two sets of bonus rules and let S and $\tilde{S}$ be the corresponding bonus systems with $\bar{Q}$-optimal bonus scales. Then $\bar{Q}(S) < \bar{Q}(\tilde{S})$ if and only if $\bar{e}(R) > \bar{e}(\tilde{R})$.

Relation (2.2) in Norberg (1976) is readily generalized as follows.

Theorem 3. The equalities

$$E\,\pi_R^{(n)}(Z_{R,n}) = EX_n , \qquad n = 0,1,\ldots \tag{4.10}$$

and

$$\sum_{n=0}^{\infty} w_n E\,\bar{\pi}_R(Z_{R,n}) = \sum_{n=0}^{\infty} w_n\, E\,X_n \tag{4.11}$$

are valid for any bonus rules R .

Proof: Relation (4.10) follows from the identities $E\,E(E_\theta X_n | Z_{R,n}) = E\,E_\theta X_n = E\,X_n$. Likewise, $E\,\bar{\pi}_R(Z_{R,N}) = E\,E_\theta X_N$ which by iterated expectations is seen to be equivalent to (4.11). □

The relations (4.10)and (4.11) actually are equivalent. We obtain (4.10) from (4.11) by letting $w_n = 1$. On the other hand, the left hand side of (4.11) may easily be rewritten as $\Sigma_{n=0}^{\infty} w_n\, E\pi_R^{(n)}(Z_{R,n})$, so that (4.11) follows from (4.10).

5. Discussion and numerical examples

The criterion (4.5) enables us to discuss various short term aspects which were excluded from the analysis based on the purely asymptotic criterion. We demonstrate this by working out two examples based on the 1975 Norwegian bonus system described by Norberg (1976, Section 1), in a manner which permits a comparison with his results. The transition rules of the system are given by the following matrix.

$$\underset{\sim}{T} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \end{matrix} \\ \begin{bmatrix} \{1,2,\ldots\} & & & \{0\} & & & & & & & & & \\ \{1,2,\ldots\} & & & \{0\} & & & & & & & & & \\ \{1,2,\ldots\} & & & \{0\} & & & & & & & & & \\ \{2,3,\ldots\} & \{1\} & & & \{0\} & & & & & & & & \\ \{2,3,\ldots\} & & \{1\} & & & \{0\} & & & & & & & \\ \{3,4,\ldots\} & \{2\} & & \{1\} & & & \{0\} & & & & & & \\ \{3,4,\ldots\} & & \{2\} & & \{1\} & & & \{0\} & & & & & \\ \{4,5,\ldots\} & \{3\} & & \{2\} & & \{1\} & & & \{0\} & & & & \\ \{4,5,\ldots\} & & \{3\} & & \{2\} & & \{1\} & & & \{0\} & & & \\ \{5,6,\ldots\} & \{4\} & & \{3\} & & \{2\} & & \{1\} & & & \{0\} & & \\ \{5,6,\ldots\} & & \{4\} & & \{3\} & & \{2\} & & \{1\} & & & \{0\} & \\ \{6,7,\ldots\} & \{5\} & & \{4\} & & \{3\} & & \{2\} & & \{1\} & & & \{0\} \\ \{6,7,\ldots\} & & \{5\} & & \{4\} & & \{3\} & & \{2\} & & \{1\} & & \{0\} \end{bmatrix} \end{array} \begin{matrix} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \\ 10 \\ 11 \\ 12 \\ 13 \end{matrix} \qquad (5.1)$$

Example 1. The choice of an initial class. As the number of insurance periods increases, the influence of the initial class diminishes for the individual policy, and it vanishes in the limit. Therefore, the choice of the initial class cannot be made part of the optimizing procedure based on the asymptotic measure Q_0. Norberg (1976) suggested that the initial class k might be chosen so as to minimize the rating error in the initial period,

$|E X_1 - \pi_R^{(0)}(k)|$. We will now investigate another possibility.

When the general criterion (4.5) is used with nonzero weights $w_1, w_2, \ldots$, $\bar{Q}(S)$ effectively depends on k , and (4.9) measures the efficiency of $\underset{\sim}{T}$ as well as of k . Thus to any $\underset{\sim}{T}$ we can find an optimal initial class, $k_{\underset{\sim}{T}}$.

Let us adopt the model used in Norberg (1976). Conditionally, given $\Theta = \theta$, claims are generated by a homogeneous compound Poisson process with claim intensity θ . The claim numbers $M_1, M_2, \ldots$ are independent and identically distributed (i.i.d.) according to the Poisson distribution with parameter θ , and the severities Y_{nj} , $n = 1,2,\ldots$; $j = 1,2,\ldots$, are also i.i.d. and independent of $M_1, M_2, \ldots$. Moreover, the severities are assumed to be independent of Θ . We put $EY = 1$, which means that EY is chosen as monetary unit. For fixed $\Theta = \theta$, $\{Z_{R,n}\}_{n=1}^{\infty}$ is a homogeneous Markov chain on the bonus classes, with one-step transition probability matrix

$$\underset{\sim}{P}_{\underset{\sim}{T},\theta} = \{p_{\underset{\sim}{T},\theta}(i,j)\}_{i,j=1}^{K} \tag{5.2}$$

given by

$$p_{\underset{\sim}{T},\theta}(i,j) = P_\theta(Z_{R,n+1} = j \mid Z_{R,n} = i) = P_\theta(M_n \in T_{ij}) = \sum_{r \in T_{ij}} \frac{\theta^r}{r!} e^{-\theta} ,$$
$i,j = 1,\ldots,K$.

With the transition rules (5.1) we have, for instance, $p_{\underset{\sim}{T},\theta}(5,6) = e^{-\theta}$, $p_{\underset{\sim}{T},\theta}(5,3) = \theta e^{-\theta}$, and $p_{\underset{\sim}{T},\theta}(5,1) = 1 - \theta e^{-\theta} - e^{-\theta}$. The conditional distribution (3.1) can be calculated recursively from the relations

$$\underset{\sim}{p}_{R,\theta}^{(n)} = \underset{\sim}{p}_{R,\theta}^{(n-1)} \underset{\sim}{P}_{\underset{\sim}{T},\theta} = \underset{\sim}{e}_k \underset{\sim}{P}_{\underset{\sim}{T},\theta}^{n-1} , \qquad n = 2,3,\ldots , \tag{5.3}$$

where $\underset{\sim}{e}_k = \underset{\sim}{p}_{R,}^{(1)}$ is the vector with 1 in the k-th entry and zero elsewhere, representing the certain event $Z_{R,1} = k$.

We also adopt the structural distribution given in Table 2 in Norberg (1976). It represents a discrete approximation to the actual distribution $U(\cdot)$ in a typical Norwegian portfolio of insured private passenger automobiles.

The $\bar{Q}$-optimal bonus scale and the efficiency $\bar{e}(R)$ can be calculated by the formulas established in Section 4.

Tables 1A to 1C below exhibit the results for three different sets of weights $\{w_n\}$. Table 1A refers to uniform weights 1/20 over the range $n = 1,\dots,20$. Tables 1B and 1C refer to geometrically decreasing weights w_n proportional to v^{n-1} , $v = 1/1.05$ and $v = 1/1.15$, respectively, for $n = 1,\dots,20$. In all tables $w_n = 0$ for $n > 20$. From the efficiencies given in the rightmost column one can see that $k = 10$ is an optimal initial class for the first two sets of weights, and $k = 11$ is optimal for the last two sets. In the Norwegian system $k = 6$. We may conclude that the Norwegian bonus rules could be improved by allotting newcomers to a higher bonus class than in the system of 1975 (and adjusting the scale accordingly) if the hypothesis of homogeneity in risk over time holds true. □

The proviso of the final remark in Example 1 is the stepping--stone to the theme of our next example.

Example 2. Time heterogeneity. The hypothesis of time homogeneity, which was crucial for the practical validity of the results based on the asymptotic criterion, is no longer required when we work with the criterion (4.5). This is an advantage, for real life risk statistics (e.g. Lemaire, 1977) show that the accident proneness of young, unexperienced drivers by far exceeds the average level. Among non-life actuaries this phenomenon is called "the duration effect".

To illustrate this effect, we consider a simple modification of the model used in Example 1. All model elements are retained, except that the parameter θ is magnified by a factor $a_n > 1$ for the early n. The calculation of $\bar{Q}$-optimal premiums and efficiencies is performed as in Example 1, with the obvious modification that (5.2) must be replaced by $\underset{\sim}{P}^{[n]}_{\underset{\sim}{T},\theta} = \underset{\sim}{P}_{\underset{\sim}{T},a_n\theta}$, $n = 1,2,\ldots,$ and (5.3) by $\underset{\sim}{p}^{(n)}_{R,\theta} = \underset{\sim}{e}_k \underset{\sim}{P}^{[1]}_{\underset{\sim}{T},\theta} \cdots \underset{\sim}{P}^{[n-1]}_{\underset{\sim}{T},\theta}$.

As a first example we have chosen duration effects $a_1 = 3$, $a_2 = 2.5$, $a_3 = 2$, $a_4 = 1.75$, $a_5 = 1.5$, $a_6 = 1.25$, $a_7 = 1.15$, $a_8 = 1.05$, and $a_n = 1$ for $n > 8$.

The corresponding premiums and efficiencies are displayed in Table 2A, for the uniform weights of Table 1A; and in Table 2B for the decreasing weights of Table 1C. A comparison of Table 2A with Table 1A and of Table 2B with Table 1C shows that the duration effect, as specified by us, yields an increase in the premium level. This was expected, of course, since we have taken all $a_n \geq 1$ and thereby increased the accident proneness level for all risks in the portfolio. It is more interesting to note that it is optimal to take class 1 as the initial class in the presence of the specified duration effect.

Similar calculations have been performed also with the less drastic duration factors $a_1 = 2$, $a_2 = 1.6$, $a_3 = 1.4$, $a_4 = 1.25$, $a_5 = 1.15$, $a_6 = 1.1$, $a_7 = 1.05$, and $a_n = 1$ for $n > 7$. The results are presented in Tables 2C and 2D. As could be expected, a comparison with the results in Tables 2A and 2B shows that the premium level and the optimal initial class are less affected by these more moderately sized duration factors than by those considered above. $\square$

The weights $\{w_n\}$ may be given various interpretations. We will mention two of these. Let $v = 1/(1+i)$ be the discount factor at rate of interest i . One possibility is to let w_n be proportional to v^{n-1} , viz., to let $w_n = iv^n$ for all $n \geq 1$, while $w_0 = 0$. With these weights, relation (4.11) states that for new policies the present value of future premiums balances the present value of future claims on the average. This assumes that premiums are paid at the beginning of the year and that X_n is the claim amount in year n discounted to the beginning of that year.

A second possibility is to let $\{w_n\}$ represent the age distribution of the policies, assumed to be independent of calendar time. In this case, the random variable N has an obvious and appealing interpretation; it is the age of a policy picked at random from the portfolio. The premium in (4.8) now becomes optimal in a very reasonable sense, for it is designed so as to minimize the expected squared rating error for a randomly chosen policy. Relation (4.11) now states that on the average premiums and claims balance for the portfolio as a whole. This desirable property is not achieved in general for other choices of weights $\{w_n\}$, such as for the discounting factors mentioned above.

The choice of $\{w_n\}$ is, of course, of vital importance only if the optimal bonus elements are sensitive to changes in these weights. By inspection of Tables 1A to 1C we see that when the transition rules are as in (5.1) and the starting class is fixed, the optimal bonus scale is fairly robust to moderate changes in the weights. From (4.6) it is seen that the $\bar{Q}$-optimal class premuim $\bar{\pi}_R(j)$ is a weighted average of those Q_n-optimal class premiums $\pi_R^{(n)}(j)$ for which $w_n > 0$. (The weights depend on the class j.) Hence we can get an impression of the sensitivity of $\bar{\pi}_R$ to changes in $\{w_n\}$ by studying how $\pi_R^{(n)}$ changes with increasing n. This can be done by inspection of Tables 3A and 3B, which are based on the model assumptions of Example 1, with starting classes 6 and 11 respectively. The rows of these tables are the vectors $\pi^{(n)}$ for selected n. Empty entries (n,j) signify that $P(Z_n = j) = 0$, i.e. class j cannot be reached in n steps under the bonus rules considered.

In Table 3A, which refers to the case where the starting class is $k = 6$, as in the 1975 system in Norway, the Q_n-optimal $\pi_R^{(n)}$ is seen to converge slowly to the asymptotically optimal scale $\pi_R^{(0)}$. Moreover, for all j, the class premium $\pi_R^{(n)}(j)$ shows an upward, though not strictly monotonic, trend as n increases. At first sight this might seem surprising as it appears to imply that the total premium level increases with increasing n, which it should not in the time-homogeneous case. This apparent paradox is resolved once we realize that the distribution of the policies over the bonus classes changes from year to year, with increasing proportions in higher classes.

Table 3B refers to the case where new policies start in class 11. First note that $\pi_R^{(n)}$ now converges more quickly to $\pi_R^{(0)}$ than when $k = 6$. This could be expected since asymptotically the majority of the policies are found in the higher bonus classes. Roughly spoken, the initial distribution concentrated in $k = 11$ fits the asymptotic distribution better than does the one concentrated in

$k = 6$. The last remark also explains why we do not find the same kind of upward trend in all class premiums as was found in Table 3A. Yet another effect that could be anticipated from the present discussion is that the $\bar{Q}$-optimal bonus scale is less sensitive to changes in $\{w_n\}$ when $k = 11$ than when $k = 6$. This conjecture is supported by Tables 1A to 1C. In fact, $\bar{\pi}_R$ is remarkably stable when $k = 11$.

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Tables

Table 1 A $\bar{Q}$-optimal premiums[a)] for each bonus class j , by starting class k , for weights $w_1 = \cdots = w_{20} = 1/20$ and $w_n = 0$ for $n > 20$.

k \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	$\bar{e}(R)$
1	0.189	0.250	0.237	0.193	0.184	0.175	0.166	0.158	0.151	0.143	0.134	0.126	0.098	0.02430
2	0.277	0.174	0.237	0.193	0.184	0.175	0.166	0.158	0.151	0.143	0.134	0.126	0.098	0.02443
3	0.277	0.250	0.170	0.193	0.184	0.175	0.166	0.158	0.151	0.143	0.134	0.126	0.098	0.02446
4	0.313	0.253	0.240	0.195	0.186	0.177	0.169	0.161	0.154	0.146	0.137	0.128	0.100	0.02466
5	0.326	0.290	0.244	0.254	0.189	0.180	0.172	0.164	0.158	0.150	0.140	0.132	0.102	0.02496
6	0.347	0.301	0.279	0.268	0.244	0.184	0.176	0.169	0.162	0.155	0.145	0.136	0.105	0.02517
7	0.360	0.322	0.290	0.295	0.258	0.237	0.181	0.174	0.167	0.160	0.150	0.141	0.108	0.02532
8	0.376	0.334	0.310	0.308	0.283	0.251	0.231	0.180	0.173	0.165	0.154	0.146	0.111	0.02542
9	0.388	0.349	0.321	0.325	0.295	0.274	0.245	0.228	0.179	0.171	0.159	0.150	0.114	0.02547
<u>10</u>	0.399	0.359	0.333	0.336	0.309	0.285	0.264	0.240	0.219	0.177	0.165	0.155	0.117	0.02549
11	0.407	0.368	0.341	0.345	0.317	0.296	0.272	0.256	0.228	0.214	0.170	0.160	0.120	0.02548
12	0.413	0.374	0.346	0.350	0.322	0.301	0.278	0.262	0.235	0.220	0.183	0.165	0.122	0.02539
13	0.415	0.376	0.349	0.353	0.324	0.303	0.280	0.264	0.237	0.224	0.185	0.176	0.125	0.02530

a) Tables 1 A to 1 C show $\bar{\pi}_R(j)$ for $j = 1,\dots,13$ and $\bar{e}(R)$ corresponding to the transition rules $\underline{T}$ given by (5.1), starting classes $k = 1,\dots,13$ and three different sets of weights $\{w_n\}$. The model assumptions are those of Example 1. The $\bar{Q}$-optimal starting class is underlined in each of the tables.

Table 1 B $\bar{Q}$-optimal premiums[a)] for each bonus class j, by starting class k, for weights w_n proportional to v^{n-1} for $n = 1,\ldots,20$, with $v = 1/1.05$, and $w_n = 0$ for $n > 20$.

k \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	$\bar{e}(R)$
1	0.180	0.240	0.229	0.183	0.174	0.165	0.158	0.151	0.144	0.137	0.129	0.122	0.097	0.02399
2	0.265	0.168	0.229	0.183	0.174	0.165	0.158	0.151	0.144	0.137	0.129	0.122	0.097	0.02413
3	0.265	0.240	0.164	0.183	0.174	0.165	0.158	0.151	0.144	0.137	0.129	0.122	0.097	0.02417
4	0.305	0.243	0.231	0.184	0.175	0.167	0.160	0.153	0.146	0.139	0.132	0.124	0.099	0.02437
5	0.317	0.283	0.235	0.245	0.178	0.170	0.163	0.156	0.150	0.143	0.135	0.128	0.101	0.02466
6	0.342	0.293	0.273	0.258	0.235	0.175	0.167	0.160	0.154	0.148	0.140	0.132	0.104	0.02487
7	0.355	0.317	0.284	0.288	0.249	0.229	0.173	0.166	0.159	0.152	0.145	0.136	0.107	0.02502
8	0.372	0.330	0.306	0.302	0.277	0.243	0.225	0.172	0.165	0.158	0.150	0.141	0.110	0.02511
9	0.385	0.346	0.319	0.321	0.290	0.269	0.238	0.222	0.172	0.164	0.155	0.146	0.114	0.02517
10	0.398	0.358	0.332	0.333	0.307	0.282	0.262	0.235	0.216	0.171	0.161	0.152	0.117	0.02520
11	0.407	0.368	0.341	0.344	0.316	0.295	0.271	0.255	0.226	0.212	0.167	0.157	0.120	0.02520
12	0.414	0.375	0.347	0.350	0.322	0.302	0.278	0.263	0.236	0.220	0.184	0.163	0.124	0.02512
13	0.416	0.377	0.350	0.353	0.324	0.304	0.281	0.266	0.239	0.226	0.187	0.178	0.128	0.02502

a) See footnote of Table 1 A.

Table 1 C $\bar{Q}$-optimal premiums[a)] for each bonus class j, by starting class k, for weights w_n proportional to v^{n-1} for $n = 1,\ldots,20$, with $v = 1/1.15$, and $w_n = 0$ for $n > 20$.

k \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	$\bar{e}(R)$
1	0.170	0.227	0.216	0.169	0.161	0.153	0.146	0.139	0.134	0.128	0.122	0.115	0.096	0.02361
2	0.250	0.161	0.216	0.169	0.161	0.153	0.146	0.139	0.134	0.128	0.122	0.115	0.096	0.02377
3	0.250	0.227	0.158	0.169	0.161	0.153	0.146	0.139	0.134	0.128	0.122	0.115	0.096	0.02381
4	0.293	0.230	0.219	0.172	0.163	0.155	0.148	0.141	0.135	0.129	0.124	0.117	0.097	0.02399
5	0.304	0.272	0.224	0.231	0.167	0.159	0.151	0.145	0.139	0.133	0.127	0.120	0.099	0.02426
6	0.332	0.282	0.263	0.244	0.222	0.165	0.157	0.150	0.143	0.137	0.132	0.124	0.103	0.02443
7	0.346	0.309	0.275	0.278	0.235	0.217	0.163	0.156	0.149	0.142	0.137	0.129	0.106	0.02455
8	0.367	0.322	0.300	0.292	0.268	0.231	0.215	0.163	0.156	0.148	0.142	0.134	0.110	0.02462
9	0.380	0.342	0.314	0.314	0.282	0.262	0.228	0.213	0.163	0.155	0.148	0.140	0.114	0.02466
10	0.396	0.355	0.330	0.328	0.302	0.276	0.257	0.227	0.211	0.163	0.155	0.146	0.118	0.02469
11	0.407	0.368	0.342	0.342	0.313	0.294	0.269	0.253	0.222	0.208	0.162	0.152	0.122	0.02470
12	0.415	0.376	0.349	0.351	0.322	0.302	0.280	0.263	0.238	0.218	0.187	0.159	0.127	0.02464
13	0.419	0.379	0.354	0.354	0.326	0.306	0.284	0.269	0.242	0.229	0.191	0.181	0.132	0.02453

a) See footnote of Table 1 A.

Table 2 A $\bar{Q}$-optimal premiums[a)] for each bonus class j, by starting class k, for weights $w_1 = \cdots = w_{20} = 1/20$ and $w_n = 0$ for $n > 20$, and duration factors $a_1 = 3$, $a_2 = 2.5$, $a_3 = 2$, $a_4 = 1.75$, $a_5 = 1.5$, $a_6 = 1.25$, $a_7 = 1.15$, $a_8 = 1.05$, and $a_n = 1$ for $n > 8$.

k \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	$\bar{e}(R)$
1	0.442	0.307	0.260	0.291	0.230	0.198	0.175	0.158	0.147	0.137	0.126	0.118	0.091	0.05016
2	0.431	0.407	0.260	0.291	0.230	0.198	0.175	0.158	0.147	0.137	0.126	0.118	0.091	0.04983
3	0.431	0.307	0.405	0.291	0.230	0.198	0.175	0.158	0.147	0.137	0.126	0.118	0.091	0.04970
4	0.427	0.374	0.284	0.356	0.265	0.212	0.185	0.166	0.150	0.139	0.127	0.118	0.091	0.04944
5	0.436	0.325	0.366	0.301	0.341	0.253	0.202	0.177	0.158	0.143	0.130	0.121	0.091	0.04928
6	0.425	0.388	0.306	0.336	0.275	0.338	0.248	0.196	0.172	0.153	0.134	0.124	0.093	0.04918
7	0.434	0.337	0.381	0.319	0.315	0.262	0.337	0.245	0.191	0.167	0.143	0.129	0.095	0.04912
8	0.423	0.391	0.321	0.349	0.297	0.308	0.256	0.337	0.240	0.188	0.155	0.139	0.098	0.04898
9	0.432	0.347	0.384	0.332	0.330	0.286	0.304	0.252	0.330	0.238	0.171	0.152	0.102	0.04867
10	0.423	0.392	0.333	0.356	0.312	0.324	0.279	0.301	0.242	0.329	0.210	0.170	0.108	0.04816
11	0.433	0.357	0.384	0.342	0.338	0.302	0.316	0.274	0.285	0.237	0.281	0.212	0.117	0.04681
12	0.426	0.394	0.342	0.362	0.322	0.329	0.291	0.309	0.256	0.277	0.200	0.288	0.132	0.04575
13	0.433	0.363	0.382	0.347	0.342	0.305	0.317	0.275	0.286	0.239	0.227	0.189	0.160	0.04282

a) Tables 2 A to 2 D show $\bar{\pi}_R(j)$ for $j = 1,\ldots,13$ and $\bar{e}(R)$ corresponding to the transition rules $\underline{T}$ given by (5.1), starting classes $k = 1,\ldots,13$ and various sets of weights $\{w_n\}$ and duration factors $\{a_n\}$, under the model assumptions of Example 2. The $\bar{Q}$-optimal starting class is underlined in each case.

Table 2 B $\bar{Q}$-optimal premiums[a)] for each bonus class j, by starting class k, for weights w_n proportional to v^{n-1} for $n = 1,\ldots,20$, with $v = 1/1.15$, and $w_n = 0$ for $n > 20$, and duration factors $\{a_n\}$ as in Table 2 A.

	1	2	3	4	5	6	7	8	9	10	11	12	13	$\bar{e}(R)$
1	0.456	0.314	0.256	0.301	0.226	0.187	0.159	0.139	0.128	0.119	0.112	0.106	0.087	0.08237
2	0.458	0.428	0.256	0.301	0.226	0.187	0.159	0.139	0.128	0.119	0.112	0.106	0.087	0.08186
3	0.458	0.314	0.431	0.301	0.226	0.187	0.159	0.139	0.128	0.119	0.112	0.106	0.087	0.08173
4	0.464	0.405	0.294	0.392	0.279	0.208	0.174	0.150	0.131	0.121	0.113	0.105	0.087	0.08095
5	0.476	0.340	0.405	0.314	0.387	0.271	0.200	0.168	0.144	0.125	0.116	0.107	0.087	0.08052
6	0.467	0.436	0.324	0.370	0.286	0.388	0.269	0.197	0.165	0.140	0.121	0.111	0.088	0.08023
7	0.481	0.356	0.438	0.339	0.354	0.276	0.390	0.269	0.195	0.163	0.136	0.118	0.091	0.08011
8	0.468	0.445	0.343	0.393	0.316	0.354	0.274	0.391	0.268	0.194	0.157	0.134	0.096	0.07985
9	0.482	0.366	0.448	0.355	0.380	0.309	0.354	0.273	0.390	0.268	0.186	0.156	0.103	0.07956
10	0.469	0.448	0.355	0.402	0.335	0.380	0.307	0.355	0.268	0.390	0.252	0.187	0.115	0.07896
11	0.483	0.376	0.449	0.365	0.389	0.328	0.378	0.305	0.345	0.266	0.365	0.256	0.132	0.07722
12	0.470	0.448	0.364	0.405	0.345	0.387	0.322	0.375	0.291	0.341	0.234	0.373	0.166	0.07546
13	0.480	0.378	0.443	0.367	0.389	0.327	0.377	0.303	0.351	0.270	0.288	0.219	0.232	0.06875

a) See footnote of Table 2 A.

Table 2 C $\bar{Q}$-optimal premiums[a)] for each bonus class j, by starting class k, for weights $w_1 = \cdots = w_{20} = 1/20$ and $w_n = 0$ for $n > 20$, and duration factors $a_1 = 2$, $a_2 = 1.6$, $a_3 = 1.4$, $a_4 = 1.25$, $a_5 = 1.15$, $a_6 = 1.1$, $a_7 = 1.05$, and $a_n = 1$ for $n > 7$.

k \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	$\bar{e}(R)$
1	0.314	0.272	0.245	0.237	0.206	0.185	0.170	0.159	0.149	0.140	0.130	0.122	0.095	0.03328
2	0.330	0.294	0.245	0.237	0.206	0.185	0.170	0.159	0.149	0.140	0.130	0.122	0.095	0.03328
3	0.330	0.272	0.289	0.237	0.206	0.185	0.170	0.159	0.149	0.140	0.130	0.122	0.095	0.03325
4	0.344	0.298	0.258	0.279	0.223	0.195	0.176	0.162	0.152	0.143	0.132	0.123	0.095	0.03357
5	0.353	0.297	0.290	0.270	0.270	0.215	0.188	0.170	0.157	0.147	0.135	0.126	0.097	0.03377
6	0.360	0.326	0.285	0.291	0.256	0.266	0.210	0.184	0.166	0.153	0.139	0.130	0.099	0.03395
7	0.370	0.319	0.318	0.297	0.278	0.247	0.264	0.207	0.180	0.163	0.145	0.135	0.102	0.03406
8	0.375	0.344	0.306	0.316	0.284	0.272	0.242	0.262	0.204	0.178	0.154	0.142	0.105	0.03411
9	0.386	0.339	0.334	0.319	0.304	0.275	0.266	0.238	0.257	0.201	0.166	0.151	0.108	0.03405
10	0.393	0.359	0.326	0.335	0.305	0.296	0.267	0.262	0.228	0.254	0.184	0.164	0.113	0.03388
11	0.402	0.358	0.346	0.338	0.320	0.294	0.285	0.260	0.247	0.223	0.223	0.183	0.118	0.03336
12	0.407	0.372	0.341	0.349	0.319	0.308	0.280	0.276	0.241	0.240	0.189	0.225	0.126	0.03289
13	0.412	0.369	0.352	0.348	0.327	0.302	0.290	0.266	0.252	0.228	0.200	0.181	0.141	0.03173

a) See footnote of Table 2 A.

Table 2 D $\bar{Q}$-optimal premiums[a)] for each bonus class j, by starting class k, for weights w_n proportional to v^{n-1} for $n = 1,\ldots,20$ with $v = 1/1.15$, and $w_n = 0$ for $n > 20$, and duration factors $\{a_n\}$ as in Table 2 C.

k \ j	1	2	3	4	5	6	7	8	9	10	11	12	13	$\bar{e}(R)$
1	0.311	0.263	0.230	0.227	0.191	0.167	0.150	0.140	0.131	0.123	0.117	0.111	0.092	0.04363
2	0.331	0.296	0.230	0.227	0.191	0.167	0.150	0.140	0.131	0.123	0.117	0.111	0.092	0.04363
3	0.331	0.263	0.295	0.227	0.191	0.167	0.150	0.140	0.131	0.123	0.117	0.111	0.092	0.04359
4	0.351	0.300	0.250	0.285	0.215	0.181	0.158	0.143	0.134	0.125	0.118	0.111	0.092	0.04387
5	0.362	0.293	0.296	0.264	0.280	0.209	0.176	0.153	0.139	0.129	0.121	0.113	0.093	0.04397
6	0.369	0.337	0.282	0.294	0.250	0.279	0.206	0.173	0.151	0.136	0.127	0.118	0.096	0.04409
7	0.381	0.317	0.334	0.296	0.283	0.243	0.279	0.205	0.172	0.149	0.134	0.124	0.099	0.04415
8	0.385	0.359	0.307	0.326	0.283	0.280	0.240	0.278	0.204	0.171	0.146	0.132	0.103	0.04417
9	0.398	0.340	0.355	0.320	0.316	0.277	0.278	0.239	0.277	0.203	0.166	0.145	0.108	0.04412
10	0.403	0.375	0.329	0.347	0.307	0.313	0.273	0.277	0.235	0.277	0.196	0.165	0.115	0.04397
11	0.415	0.361	0.368	0.341	0.335	0.300	0.307	0.270	0.268	0.232	0.261	0.196	0.126	0.04336
12	0.418	0.388	0.347	0.361	0.325	0.328	0.290	0.302	0.255	0.265	0.205	0.265	0.143	0.04271
13	0.425	0.374	0.373	0.354	0.343	0.310	0.313	0.278	0.279	0.242	0.226	0.196	0.176	0.04024

a) See footnote of Table 2 A.

Table 3 A   The Q_n-optimal premiums[a)] for each bonus class j and n = 1,2,...,10,15,20,...,50 , and n = 0 (the asymptotic scale[b)]) when the starting class is k = 6 .

n \ j	1	2	3	4	5	6	7	8	9	10	11	12	13
1						.151							
2	.325	.226		.207			.142						
3	.314	.253	.253	.257	.195			.133					
4	.307	.294	.240	.248	.244	.185			.125				
5	.312	.282	.280	.238	.235	.233	.175			.118			
6	.333	.276	.269	.281	.223	.223	.222	.166			.111		
7	.329	.311	.260	.271	.268	.211	.212	.212	.157			.106	
8	.336	.299	.296	.264	.256	.256	.201	.202	.202	.150			.101
9	.348	.297	.284	.298	.247	.243	.245	.192	.193	.194	.142		.096
10	.346	.323	.278	.288	.283	.233	.232	.235	.184	.185	.139	.136	.092
15	.371	.330	.307	.320	.282	.271	.263	.229	.217	.198	.172	.152	.105
20	.383	.345	.328	.326	.303	.291	.261	.250	.221	.212	.168	.155	.111
25	.391	.357	.331	.336	.315	.292	.271	.256	.227	.214	.168	.166	.112
30	.396	.362	.335	.343	.317	.297	.274	.257	.228	.213	.173	.165	.113
35	.399	.364	.339	.345	.320	.300	.275	.258	.227	.215	.173	.164	.113
40	.401	.366	.340	.347	.321	.300	.276	.259	.228	.215	.172	.165	.114
45	.402	.367	.341	.348	.322	.301	.276	.259	.228	.215	.173	.166	.114
50	.403	.367	.341	.348	.322	.301	.276	.259	.228	.215	.173	.166	.114
0	.403	.368	.342	.349	.323	.301	.276	.259	.228	.215	.173	.166	.114

a) Tables 3 A and 3 B show $\pi_R^{(n)}(j)$ corresponding to $\underset{\sim}{T}$ given by (5.1) for j = 1,2,...,13 and various n when the starting class is k = 6 and k = 11 , respectively. The model assumptions are those of Example 1.

b) Due to rounding errors the structural distribution used in Example 1 in this paper differs slightly from that originally used by Norberg (1976, Table 2). As a consequence of this the asymptotic scale given here does not equal the one in Norberg (1976, Table 3) completely.

Table 3 B The Q_n-optimal premiums[a)] for each bonus class j and n = 1,2,...,10,15,20,...,50 , and n = 0 (the asymptotic scale[b)]) when the starting class is k = 11 .

n \ j	1	2	3	4	5	6	7	8	9	10	11	12	13
1											.151		
2	.429		.374		.321		.266		.207			.142	
3	.413	.406	.355	.357	.305	.305	.253	.253		.195			.133
4	.402	.387	.338	.341	.292	.290	.290	.240	.240		.185		.125
5	.401	.370	.332	.328	.324	.277	.277	.277	.229	.229	.175	.175	.118
6	.408	.357	.353	.329	.310	.309	.265	.265	.221	.218	.176	.166	.125
7	.406	.363	.339	.350	.303	.296	.273	.254	.237	.210	.191	.167	.121
8	.404	.373	.334	.338	.323	.287	.278	.255	.236	.222	.175	.182	.118
9	.409	.363	.348	.339	.315	.301	.270	.264	.224	.225	.178	.166	.121
10	.409	.366	.340	.350	.311	.298	.272	.258	.235	.212	.188	.170	.119
15	.408	.368	.344	.347	.319	.300	.274	.261	.226	.222	.178	.167	.118
20	.407	.370	.341	.348	.322	.299	.276	.258	.230	.218	.174	.170	.116
25	.406	.369	.341	.349	.322	.300	.275	.259	.230	.215	.175	.168	.115
30	.405	.368	.342	.349	.322	.301	.275	.259	.228	.216	.174	.166	.115
35	.404	.368	.342	.349	.322	.301	.276	.259	.228	.216	.173	.166	.114
40	.404	.368	.342	.349	.322	.301	.276	.259	.228	.215	.173	.166	.114
45	.404	.368	.342	.349	.323	.301	.276	.259	.228	.215	.173	.166	.114
50	.404	.368	.342	.349	.323	.301	.276	.259	.228	.215	.173	.166	.114
0	.403	.368	.342	.349	.323	.301	.276	.259	.228	.215	.173	.166	.114

a) See footnote of Table 3 A.

b) See footnote of Table 3 A.